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2－一致凸 Banach 空间的特征不等式*

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摘 要：用 Banach 空间理论的方法研究了 2－一致凸空间的特征不等式问题，作为徐总本和 Roach 得到的一致凸 Banach 空间的特征不等式之推广给出了 2－一致凸 Banach 空间的特征不等式。

关键词：2－一致凸 Banach 空间；2 维凸性模；特征不等式

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The characteristic inequality of 2-uniformly rotund Banach spaces

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Abstract: The problem of the characteristic inequality in 2-uniformly convex spaces is discussed by using the method of Banach space theory. The characteristic inequality of the 2-uniformly rotund Banach spaces is given as a generalization of the characteristic inequality in the uniformly rotund Banach spaces which was given by Zong-Ben Xu and Roach.

Key words: 2-uniformly rotund Banach space ; 2-modulus of rotundity; characteristic inequality

In 1936, the concept of a uniformly rotund Banach space was first introduced by Clarkson[1], and this class of Banach spaces is very interesting and has numerous applications. Consequently, some methods were found to investigate the geometry of Banach space (see [1] ~ [11]). In 1977, Sullivan[2] introduced the 2-uniformly rotund spaces as a generalization of uniformly rotund Banach spaces.

In this paper, X will denotes a real Banach space and X^* will denotes its dual space, symbols

$$U(X) = \{x : x \in X, \|x\| \leqslant 1\},$$
$$S(X) = \{x : x \in X, \|x\| = 1\}$$

denote the unit ball and the unit sphere in X respectively. For arbitrarily real numbers $\lambda_1, \lambda_2, \lambda_3$, we always let $\lambda_1 \vee \lambda_2 \vee \lambda_3 = \max(\lambda_1, \lambda_2, \lambda_3), \lambda \wedge \lambda_2 \wedge \lambda_3 = \min(\lambda_1, \lambda_2, \lambda_3)$, and for all $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$ are always assumed to be such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

For an arbitrary space X, one of the measuring the "2-uniformly" of the set of three dimensional subspaces is in terms of the real valued modulus of rotundity, i. e. for $\varepsilon > 0$,

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x_1 + x_2 + x_3\|}{3} : x_1, x_2, x_3 \in S(X), A(x_1, x_2, x_3) \geqslant \varepsilon\right\}$$

Where

$$A(x_1, x_2, x_3) =$$

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$$\sup\left\{\begin{vmatrix}1 & 1 & 1\\ f_1(x_1) & f_1(x_2) & f_1(x_3)\\ f_2(x_1) & f_2(x_2) & f_2(x_3)\end{vmatrix};\ f_i\in S(X^*),i=1,2\right\}$$

Banach space X is said to be 2-uniformly rotund[2] if for any $\varepsilon>0$, there exists a $\delta>0$, such that for $x_1,x_2,x_3\in S(X)$, if $\|x_1+x_2+x_3\|>3-\delta$, then $A(x_1,x_2,x_3)<\varepsilon$.

In 1989, Zongben Xu and G. F. Roach [3] gave the characteristic inequality in the uniformly rotund Banach spaces as follows: X is uniformly rotund Banach space if and only if for $\forall p\in(0,1)$, there exists a strictly increasing function $\delta_p(\lambda,\mu,\cdot):\mathbf{R}^+\to\mathbf{R}^+$, $\delta_p(\lambda,\mu,0)=0$, such that

$$\|\lambda x+\mu y\|^p+(\|x\|\vee\|y\|)^p\cdot\delta_p\left(\lambda,\mu,\frac{\|x-y\|}{\|x\|\vee\|y\|}\right)\leqslant\lambda\|x\|^p+\mu\|y\|^p,\ \forall x,y\in X$$

Where the symbol $\|x\|\vee\|y\|$ means maximum of $\|x\|$ and $\|y\|$, and $\forall\lambda,\mu\in[0,1]$ are satisfy that $\lambda+\mu=1$.

The generalization of above characteristic inequality to the 2-uniformly rotund Banach spaces which we shall consider can be motivated by the following restatement of the characteristic inequality in the uniformly rotund Banach spaces: X is uniformly rotund Banach space if and only if for $\forall p\in(0,1)$, there exists a strictly increasing function $\delta_p(\lambda,\mu,\cdot):\mathbf{R}^+\to\mathbf{R}^+$, $\delta_p(\lambda,\mu,0)=0$, such that

$$\|\lambda x+\mu y\|^p+(\|x\|\vee\|y\|)^p\cdot\delta_p\left(\lambda,\mu,\frac{A(x,y)}{\|x\|\vee\|y\|}\right)\leqslant\lambda\|x\|^p+\mu\|y\|^p,\ \forall x,y\in X$$

Where

$$A(x,y)=\sup_{f\in S(X^*)}\left\{\begin{vmatrix}1 & 1\\ f(x) & f(y)\end{vmatrix}\right\}=\sup_{f\in S(X^*)}|f(x)-f(x)|=\|x-y\|$$

Now we give the characteristic inequality in the 2-uniformly rotund Banach spaces as follows: X is 2-uniformly rotund if and only if for each $\forall p\in(0,1)$, there exists a strictly increasing function $\delta_p(\lambda_1,\lambda_2,\lambda_3,\cdot):\mathbf{R}^+\to\mathbf{R}^+$, $\delta_p(\lambda_1,\lambda_2,\lambda_3,0)=0$, such that

$$\|\sum_{i=1}^3\lambda_ix_i\|^p+(\|x_1\|\vee\|x_2\|\vee\|x_3\|)^p\cdot\delta_p\left(\lambda_1,\lambda_2,\lambda_3,\frac{A(x_1,x_2,x_3)}{(\|x_1\|\vee\|x_2\|\vee\|x_3\|)^2}\right)\leqslant\sum_{i=1}^3\lambda_i\|x_i\|^p,\ \forall x_1,x_2,x_3\in X$$

The characteristic inequality of 2-uniformly rotund Banach spaces.

Theorem 1 X is 2-uniformly rotund if and only if for each $\forall p\in(0,1)$, there exists a strictly increasing function $\delta_p(\lambda_1,\lambda_2,\lambda_3,\cdot):\mathbf{R}^+\to\mathbf{R}^+$, $\delta_p(\lambda_1,\lambda_2,\lambda_3,0)=0$, such that

$$\|\sum_{i=1}^3\lambda_ix_i\|^p+(\|x_1\|\vee\|x_2\|\vee\|x_3\|)^p\cdot\delta_p\left(\lambda_1,\lambda_2,\lambda_3,\frac{A(x_1,x_2,x_3)}{(\|x_1\|\vee\|x_2\|\vee\|x_3\|)^2}\right)\leqslant\sum_{i=1}^3\lambda_i\|x_i\|^p,\ \forall x_1,x_2,x_3\in X \quad (1)$$

In order to prove theorem 1, we give two lemmas.

Lemma 1[4] X is 2-uniformly rotund Banach space if and only if $\delta_X(\varepsilon)>0$.

Lemma 2 For $x_1,x_2,x_3\in S(X)$, $t_1,t_2\in(0,1]$, let $\varepsilon=A(x_1,x_2,x_3)\neq0$, then

$$\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3t_2x_3\|\leqslant\lambda_1+\lambda_2t_1+\lambda_3t_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_1t_2\delta_X(\varepsilon)$$

Proof (Ⅰ) Suppose that x_1,x_2,x_3 are linearly independent and denote by E the subspace spanned by the elements x_1,x_2,x_3 and the zero element, then the element $\lambda_1x_1+\lambda_2t_1x_2+\lambda_3t_2x_3$ belongs to E. Let z be the intersection point of the vector $\lambda_1x_1+\lambda_2t_1x_2-x_3$ and the ray $\tau\cdot(\lambda_1x_1+\lambda_2t_1x_2+\lambda_3t_2x_3)$ in the subspace E, where $\tau\geqslant0$. Then there exist real numbers α,β such that

$$z=\alpha(\lambda_1x_1+\lambda_2t_1x_2+\lambda_3t_2x_3),\alpha\geqslant0,$$
$$z=\beta(\lambda_1x_1+\lambda_2t_1x_2)+(1-\beta)(\lambda_1x_1+\lambda_2t_1x_2+\lambda_3x_3),\ 0\leqslant\beta\leqslant1$$

Since x_1,x_2,x_3 are linearly independent, it follows that $\alpha=1,\beta=1-t_2$, and

$$\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3t_2x_3\|=\|\beta(\lambda_1x_1+\lambda_2t_1x_2)+(1-\beta)\cdot(\lambda_1x_1+\lambda_2t_1x_2+\lambda_3x_3)\|\leqslant(1-t_2)\lambda_1+(1-t_2)\lambda_2t_1+t_2\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3x_3\|$$

Let w be the intersection point of the ray $\tau\cdot(\lambda_1x_1+\lambda_3x_3+\lambda_2t_1x_2)$, (where $\tau\geqslant0$) and the vector $\lambda_1x_1+\lambda_3x_3-x_2$. Then there exist real numbers μ,ν such that

$$w = \mu(\lambda_1 x_1 + \lambda_3 x_3 + \lambda_2 t_1 x_2), \mu \geqslant 0,$$

$$w = \nu(\lambda_1 x_1 + \lambda_3 x_3) + (1-\nu)\sum_{i=1}^{3}\lambda_i x_i, 0 \leqslant \nu \leqslant 1$$

Since x_1, x_2, x_3 are linearly independent, it follows that $\mu = 1, \nu = 1 - t_1$, and

$$\|\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 x_3\| = \|\nu(\lambda_1 x_1 + \lambda_3 x_3) + (1-\nu)\sum_{i=1}^{3}\lambda_i x_i\| \leqslant (1-t_1)\lambda_1 + (1-t_1)\lambda_3 + t_1\|\sum_{i=1}^{3}\lambda_i x_i\|$$

Therefore

$$\|\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3\| \leqslant (1-t_2)\lambda_1 + (1-t_2)\lambda_2 t_1 + t_2\|\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 x_3\| \leqslant (1-t_2)\lambda_1 + (1-t_2)\lambda_2 t_1 + (1-t_1)t_2\lambda_1 + (1-t_1)t_2\lambda_3 + t_1 t_2\|\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 x_3\| = \lambda_1 + \lambda_2 t_1 + \lambda_3 t_2 - t_1 t_2 + t_1 t_2\|\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 x_3\|$$

We define a function

$$f(\lambda, \mu, x, x_1, x_2) = \frac{\|x + \lambda x_1 + \mu x_2\| - \|x\|}{(\lambda \wedge \mu)}$$

where the symbol $\lambda \wedge \mu$ means minimum of λ and μ with $\lambda, \mu \in [0,1], \lambda^2 + \mu^2 \neq 0$, and $x, x_1, x_2 \in X$. Without loss of generality, we may assume that $\lambda_3 = \min(\lambda_1, \lambda_2, \lambda_3)$, then

$$f(\lambda_2, \lambda_3, x_1, x_2 - x_1, x_3 - x_1) - f\left(\frac{1}{3}, \frac{1}{3}, x_1, x_2 - x_1, x_3 - x_1\right) = \frac{\|x_1 + \lambda_2(x_2 - x_1) + \lambda_3(x_3 - x_1)\| - \|x_1\|}{\lambda_3} - 3\left(\|x_1 + \frac{1}{3}(x_2 - x_1) + \frac{1}{3}(x_3 - x_1)\| - \|x_1\|\right) = \frac{\|\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3\| - \|x_1\|}{\lambda_3} - 3\left(\left\|\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3\right\| - \|x_1\|\right) = \frac{\|\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3\|}{\lambda_3} - \frac{\|x_1\|}{\lambda_3} - \|x_1 + x_2 + x_3\| + 3\|x_1\| \leqslant \left\|\left(\frac{\lambda_1}{\lambda_3} - 1\right)x_1 + \left(\frac{\lambda_2}{\lambda_3} - 1\right)x_2\right\| + 3 - \frac{1}{\lambda_3} \leqslant \frac{\lambda_1}{\lambda_3} - 1 + \frac{\lambda_2}{\lambda_3} - 1 + 3 - \frac{1}{\lambda_3} = 0$$

This shows that $f(\lambda_2, \lambda_3, x_1, x_2 - x_1, x_3 - x_1) \leqslant f\left(\frac{1}{3}, \frac{1}{3}, x_1, x_2 - x_1, x_3 - x_1\right)$.

Moreover, we have

$$\|\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3\| \leqslant 1 + \lambda_3(\|x_1 + \lambda_2(x_2 - x_1) + \lambda_3(x_3 - x_1)\| - \|x_1\|)/\lambda_3 = 1 + \lambda_3 f(\lambda_2, \lambda_3, x_1, x_2 - x_1, x_3 - x_1) \leqslant 1 + 3\lambda_3\left(\left\|\frac{1}{3}\sum_{i=1}^{3}x_i\right\| - 1\right) = 1 - 3\lambda_3\left(1 - \frac{1}{3}\left\|\sum_{i=1}^{3}x_i\right\|\right) \leqslant 1 - 3\lambda_3\delta_X(\varepsilon)$$

Consequently,

$$\|\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3\| \leqslant \lambda_1 + \lambda_2 t_1 + \lambda_3 t_2 - 3(\lambda_1 \wedge \lambda_2 \wedge \lambda_3)t_1 t_2 \delta_X(\varepsilon) \tag{2}$$

（II）Suppose that x_1, x_2, x_3 are linearly dependent. Because $A(x_1, x_2, x_3) \neq 0$, so x_1, x_2, x_3 are not all linearly dependent in pairs.

If $\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3 = 0$, then the conclusion is obviously.

If $\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3 \neq 0$, it is impossible that $\lambda_1 x_1 + \lambda_2 t_1 x_2$ and x_3 are collinear is simultaneous with $\lambda_1 x_1 + \lambda_3 t_2 x_3$ and x_2 are collinear. Otherwise, there exist real numbers, λ, μ such that

$$\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda x_3 = 0 \tag{3}$$

$$\lambda_1 x_1 + \mu x_2 + \lambda_3 t_2 x_3 = 0 \tag{4}$$

From (3) and (4), we know that x_2 and x_3 are non-collinear, it follows that $\lambda = \lambda_3 t_2$, $\mu = \lambda_2 t_1$. This is incompatible with $\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3 \neq 0$.

① When $\lambda_1 x_1 + \lambda_2 t_1 x_2$ and x_3 are non-collinear, $\lambda_1 x_1 + \lambda_3 t_2 x_3$ and x_2 are collinear, denote by E the subspace spanned by the elements x_1, x_2, x_3 and the zero element, then the element $\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3$ belongs to E. Let z be the intersection point of the vector $\lambda_1 x_1 + \lambda_2 t_1 x_2 - x_3$ and the ray $\tau \cdot (\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3)$ in the subspace E, where $\tau \geqslant 0$. Then there exist real numbers α, β such that

$$z = \alpha(\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3), \alpha \geqslant 0 \tag{5}$$

$$z = \beta(\lambda_1 x_1 + \lambda_2 t_1 x_2) + (1-\beta) \cdot (\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 x_3), 0 \leqslant \beta \leqslant 1 \tag{6}$$

(6) $\times \alpha$ − (5), we have

$$\alpha z - z = \alpha(\lambda_1 x_1 + \lambda_2 t_1 x_2 + (1-\beta)\lambda_3 x_3) - \alpha(\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3) = \alpha(1 - \beta - t_2)\lambda_3 x_3$$

Since z and x_3 are linearly independent, it follows that $\alpha = 1$, $\beta = 1 - t_2$, and

$$\|\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 t_2 x_3\| = \|\beta(\lambda_1 x_1 + \lambda_2 t_1 x_2) + (1-\beta)(\lambda_1 x_1 + \lambda_2 t_1 x_2 + \lambda_3 x_3)\| \leqslant$$

$$(1-t_2)\lambda_1+(1-t_2)\lambda_2 t_1+ t_2\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3x_3\|$$

From $\lambda_1x_1+\lambda_3t_2x_3$ and x_2 are collinear, we know that $\lambda_1x_1+\lambda_3x_3$ and x_2 are non-collinear.

Let w be the intersection point of the ray $\tau\cdot(\lambda_1x_1+\lambda_3x_3+\lambda_2t_1x_2)$, (where $\tau\geqslant 0$) and the vector $\lambda_1x_1+\lambda_3x_3-x_2$. Then there exist real numbers μ,ν such that

$$w=\mu(\lambda_1x_1+\lambda_3x_3+\lambda_2t_1x_2),\mu\geqslant 0 \quad (7)$$

$$w=\nu(\lambda_1x_1+\lambda_3x_3)+(1-\nu)\sum_{i=1}^{3}\lambda_ix_i,0\leqslant\nu\leqslant 1 \quad (8)$$

(8) $\times\mu-$ (7), we have

$$\mu w-w=\mu(\lambda_1x_1+\lambda_3x_3+(1-\nu)\lambda_2x_2)-\mu(\lambda_1x_1+\lambda_3x_3+\lambda_2t_1x_2)=\mu(1-\nu-t_1)\lambda_2x_2$$

Since x_2 and w are linearly independent, it follows that $\mu=1,\nu=1-t_1$, and

$$\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3x_3\|=\left\|\nu(\lambda_1x_1+\lambda_3x_3)+(1-\nu)\sum_{i=1}^{3}\lambda_ix_i\right\|\leqslant (1-t_1)\lambda_1+(1-t_1)\lambda_3+t_1\left\|\sum_{i=1}^{3}\lambda_ix_i\right\|$$

Therefore

$$\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3t_2x_3\|\leqslant (1-t_2)\lambda_1+(1-t_2)\lambda_2t_1+t_2\left\|\sum_{i=1}^{3}\lambda_ix_i\right\|\leqslant (1-t_2)(\lambda_1+\lambda_2t_1)+(1-t_1)t_2(\lambda_1+\lambda_3)+t_1t_2\left\|\sum_{i=1}^{3}\lambda_ix_i\right\|=\lambda_1+\lambda_2t_1+\lambda_3t_2+t_1t_2\left(\left\|\sum_{i=1}^{3}\lambda_ix_i\right\|-1\right) \quad (9)$$

By (2) we know that

$$\left\|\sum_{i=1}^{3}\lambda_ix_i\right\|\leqslant 1-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)\delta_X(\varepsilon) \quad (10)$$

Combining (9) and (10), we have

$$\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3t_2x_3\|\leqslant \lambda_1+\lambda_2t_1+\lambda_3t_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)\delta_X(\varepsilon)$$

② When $\lambda_1x_1+\lambda_2t_1x_2$ and x_3 are collinear, $\lambda_1x_1+\lambda_3t_2x_3$ and x_2 are collinear, we can prove it greatly similar to ①.

③ When $\lambda_1x_1+\lambda_2t_1x_2$ and x_3 are non-collinear, $\lambda_1x_1+\lambda_3t_2x_3$ and x_2 are non-collinear, from the process of proving (Ⅰ), it follows that

$$\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3t_2x_3\|\leqslant(1-t_2)\lambda_1+(1-t_2)\lambda_2t_1+t_2\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3x_3\|$$

Now we divide two possible cases:

Case $\otimes_1$ If $\sum_{i=1}^{3}\lambda_ix_i=0$, then

$$\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3t_2x_3\|\leqslant(1-t_2)\lambda_1+(1-t_2)\lambda_2t_1+t_2\|\lambda_1x_1+\lambda_2t_1x_2+\lambda_3x_3\|=(1-t_2)\lambda_1+(1-t_2)\lambda_2t_1+t_2(\lambda_2-\lambda_2t_1)=\lambda_1+\lambda_2t_1+\lambda_3t_2-2\lambda_2t_1t_2-(\lambda_1+\lambda_3-\lambda_2)t_2\leqslant\lambda_1+\lambda_2t_1+\lambda_3t_2-2\lambda_2t_1t_2-(\lambda_1+\lambda_3-\lambda_2)t_1t_2=\lambda_1+\lambda_2t_1+\lambda_3t_2-t_1t_2\leqslant\lambda_1+\lambda_2t_1+\lambda_3t_2-t_1t_2\delta_X(\varepsilon)\leqslant\lambda_1+\lambda_2t_1+\lambda_3t_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_1t_2\delta_X(\varepsilon)$$

Case $\otimes_2$ If $\sum_{i=1}^{3}\lambda_ix_i\neq 0$, then $\lambda_1x_1+\lambda_2x_2$ and x_3 are collinear and $\lambda_1x_1+\lambda_3x_3$ and x_2 collinear is impossible simultaneous. When $\lambda_1x_1+\lambda_2x_2$ and x_3 are non-collinear, we can prove it greatly similar to ①. When $\lambda_1x_1+\lambda_3x_3$ and x_2 collinear $\lambda_1x_1+\lambda_2x_2$ and x_3 are non-collinear, then $\lambda_1x_1+\lambda_3t_2x_3$ and x_2 non-collinear. In this case, we can prove it greatly similar to ②. Thus, we complete the proof of lemma 2.

Proof of theorem 1 When the inequality (1) is satisfied, for $\forall x_1,x_2,x_3\in S(X)$ with $A(x_1,x_2,x_3)\geqslant\varepsilon$. We take $\lambda_1=\lambda_2=\lambda_3=\frac{1}{3}$ in the inequality (1), then

$$\left\|\frac{x_1+x_2+x_3}{3}\right\|\leqslant\left(1-\delta_p\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\varepsilon\right)\right)^{\frac{1}{p}}$$

Which implies that

$$\delta_X(\varepsilon)\geqslant 1-\left(1-\delta_p\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\varepsilon\right)\right)^{\frac{1}{p}}>0$$

Thus, applying Lemma 1, we know that X is 2-uniformly rotund Banach space.

Conversely, suppose that X is a 2-uniformly rotund Banach space. We shall construct a function $\delta_p(\lambda_1,\lambda_2,\lambda_3,\varepsilon)$, so that the inequality (1) is fulfilled. For this purpose, we first define a function

$$\varphi_p(\lambda_1,\lambda_2,\lambda_3,\varepsilon)=\min\{f_p^{(1)},f_p^{(2)},f_p^{(3)},f_p^{(4)}\}$$

where

$$f_p^{(1)}(\lambda_1,\lambda_2,\lambda_3,\varepsilon)=\lambda_1+(\lambda_2+\lambda_3)\left(1-\frac{\varepsilon}{12}\right)^p-\left(\lambda_1+(\lambda_2+\lambda_3)\left(1-\frac{\varepsilon}{12}\right)\right)^p,$$

$$f_p^{(2)}(\lambda_1,\lambda_2,\lambda_3,\varepsilon)=\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)^p-\frac{\left(\lambda_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)\right)^p}{(1-\lambda_2)^p},$$

$$f_p^{(3)}(\lambda_1,\lambda_2,\lambda_3,\varepsilon)=$$
$$\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)^p+\lambda_3\left(\frac{\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)}{1-\lambda_3}\right)^p-$$
$$\left(\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)+\lambda_3\left(\frac{\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)}{1-\lambda_3}\right)\right)^p,$$
$$f_p^{(4)}(\lambda_1,\lambda_2,\lambda_3,\varepsilon)=\lambda_1+$$
$$\lambda_2\left(\frac{\lambda_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)}{1-\lambda_2}\right)^p+\lambda_3\left(1-\frac{\varepsilon}{12}\right)^p-$$
$$\left(\lambda_1+\lambda_2\left(\frac{\lambda_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)}{1-\lambda_2}\right)+\lambda_3\left(1-\frac{\varepsilon}{12}\right)\right)^p$$

Now we show the following inequality:
$$\|\lambda_1x_1+\lambda_2x_2+\lambda_3x_3\|^p+\varphi_p(\lambda_1,\lambda_2,\lambda_3,\varepsilon)\leqslant$$
$$\lambda_1\|x_1\|^p+\lambda_2\|x_2\|^p+\lambda_3\|x_3\|^p$$
with the function $\delta_p(\lambda_1,\lambda_2,\lambda_3,\varepsilon)$ holds for every $\forall x_1\in S(X),x_2,x_3\in U(X)$.

Let
$$\|x_1\|=1,\ \|x_2\|=t_1,$$
$$\|x_3\|=t_2,\ \bar{x}_2=\frac{x_2}{t_1},\ \bar{x}_2=\frac{x_3}{t_2},$$
$$A(x_1,x_2,x_3)=\varepsilon,\ A(x_1,\bar{x}_2,\bar{x}_3)=\bar{\varepsilon}$$
and consider the function g defined by
$$g(t,t')=$$
$$\lambda_1+\lambda_2t^p+\lambda_3t'^p-(\lambda_1+\lambda_2t+\lambda_3t'-$$
$$3(\lambda_1\wedge\lambda_2\wedge\lambda_3)tt'\delta_X(\bar{\varepsilon}))^p,$$
$$0\leqslant t,t'\leqslant 1$$
From Lemma 2, we have
$$g(t_1,t_2)\leqslant\lambda_1+\lambda_2t_1^p+\lambda_3t_2^p-$$
$$\|\lambda_1x_1+\lambda_2t_1\bar{x}_2+\lambda_3t_2\bar{x}_3\|^p=$$
$$\lambda_1\|x_1\|^p+\lambda_2\|x_2\|^p+$$
$$\lambda_3\|x_3\|^p-\|\lambda_1x_1+\lambda_2x_2+\lambda_3x_3\| \tag{11}$$

In what follows, we will divide four possible cases which complete the steps of proving theorem 1.

① If $t_1,t_2\leqslant 1-\frac{\varepsilon}{12}$, then, without loss of generality, we may assume that $t_1\leqslant t_2$, so we have
$$g(t_1,t_2)\geqslant\lambda_1+\lambda_2t_1^p+$$
$$\lambda_3t_2^p-(\lambda_1+\lambda_2t_1+\lambda_3t_2)^p$$
Let
$$q(t_1,t_2)=\lambda_1+\lambda_2t_1^p+$$
$$\lambda_3t_2^p-(\lambda_1+\lambda_2t_1+\lambda_3t_2)^p$$
then
$$\frac{\partial q}{\partial t_1}=p\lambda_2t_1^{p-1}-p\lambda_2(\lambda_1+\lambda_2t_1+\lambda_3t_2)^{p-1}$$

Since $t_1\leqslant t_2$, it follows that $\frac{\partial q}{\partial t_1}<0$, hence
$$g(t_1,t_2)\geqslant\lambda_1+\lambda_2t_1^p+\lambda_3t_2^p-$$
$$(\lambda_1+\lambda_2t_1+\lambda_3t_2)^p\geqslant$$
$$\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)^p+$$
$$\lambda_3t_2^p-\left(\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)+\lambda_3t_2\right)^p$$
and
$$q\left(1-\frac{\varepsilon}{12},t_2\right)=\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)^p+$$
$$\lambda_3t_2^p-\left(\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)+\lambda_3t_2\right)^p$$
Because
$$\frac{\partial q\left(1-\frac{\varepsilon}{12},t_2\right)}{\partial t_2}=p\lambda_3t_2^{p-1}-$$
$$p\lambda_3\left(\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)+\lambda_3t_2\right)^{p-1}$$
so we have $\frac{\partial q\left(1-\frac{\varepsilon}{12},t_2\right)}{\partial t_2}<0$ when $t_2\leqslant 1-\frac{\varepsilon}{12}$.

Hence
$$g(t_1,t_2)\geqslant\lambda_1+\lambda_2t_1^p+\lambda_3t_2^p-$$
$$(\lambda_1+\lambda_2t_1+\lambda_3t_2)^p\geqslant$$
$$\lambda_1+(\lambda_2+\lambda_3)\left(1-\frac{\varepsilon}{12}\right)^p-\left(\lambda_1+(\lambda_2+\lambda_3)\right.$$
$$\left.\left(1-\frac{\varepsilon}{12}\right)\right)^p=f_p^{(1)}(\lambda_1,\lambda_2,\lambda_3,\varepsilon)$$

② If $t_1,t_2>1-\frac{\varepsilon}{12}$, then, we can check the following inequality $\varepsilon\leqslant\bar{\varepsilon}+4(2-t_1-t_2)$ holds for ε and $\bar{\varepsilon}$.

Indeed,
$$\varepsilon=A(x_1,x_2,x_3)\leqslant A(x_1,\bar{x}_2,\bar{x}_3)+$$
$$\sup_{f_1,f_2\in S(X^*)}\left\{\begin{vmatrix}1&1&0\\f_1(x_1)&f_1(\bar{x}_2)&f_1(x_3-\bar{x}_3)\\f_2(x_1)&f_2(\bar{x}_2)&f_2(x_3-\bar{x}_3)\end{vmatrix}\right\}+$$
$$\sup_{f_1,f_2\in S(X^*)}\left\{\begin{vmatrix}1&0&1\\f_1(x_1)&f_1(x_2-\bar{x}_2)&f_1(x_3)\\f_2(x_1)&f_2(x_2-\bar{x}_2)&f_2(x_3)\end{vmatrix}\right\}\leqslant$$
$$A(x_1,\bar{x}_2,\bar{x}_3)+$$
$$\sup_{f_1,f_2\in S(X^*)}\left\{\begin{vmatrix}1&1&0\\f_1(x_1)&f_1(\bar{x}_2)&(t_2-1)f_1(\bar{x}_3)\\f_2(x_1)&f_2(\bar{x}_2)&(t_2-1)f_2(\bar{x}_3)\end{vmatrix}\right\}+$$

$$\sup_{f_1,f_2\in S(X^*)}\left\{\begin{vmatrix}1 & 0 & 1\\ f_1(x_1) & (t_1-1)f_1(\bar{x}_2) & f_1(\bar{x}_3)\\ f_2(x_1) & (t_1-1)f_2(\bar{x}_2) & f_2(\bar{x}_3)\end{vmatrix}\right\}\leqslant$$
$$\bar{\varepsilon}+\sup_{f_1,f_2\in S(X^*)}\{(t_2-1)(f_2(\bar{x}_3)f_1(\bar{x}_2)-f_2(\bar{x}_2)f_1(\bar{x}_3)-f_2(\bar{x}_3)f_1(x_1)+f_1(\bar{x}_3)f_2(x_1))\}+$$
$$\sup_{f_1,f_2\in S(X^*)}\{(t_2-1)(t_2f_1(\bar{x}_2)f_2(\bar{x}_3)-t_2f_2(\bar{x}_2)f_1(\bar{x}_3)+f_2(\bar{x}_2)f_1(x_1)-f_1(\bar{x}_2)f_2(x_1))\}\leqslant$$
$$\bar{\varepsilon}+4(1-t_2)+2(1-t_1)(t_2+1)\leqslant$$
$$\bar{\varepsilon}+4(2-t_1-t_2)$$

It follows that

$$\bar{\varepsilon}\geqslant\varepsilon-4(2-t_1-t_2)=\varepsilon+4(t_1+t_2-2)\geqslant\frac{\varepsilon}{3}$$

Because the function $\delta_X(\varepsilon)$ is strictly increasing in ε, so we have

$$g(t_1,t_2)\geqslant\lambda_1+\lambda_2t_1^p+\lambda_3t_2^p-$$
$$\left(\lambda_1+\lambda_2t_1+\lambda_3t_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_1t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right)^p=h(t_1,t_2)$$

From

$$\frac{\partial h}{\partial t_1}=p\lambda_2t_1^{p-1}-p\left[\lambda_1+\lambda_2t_1+\lambda_3t_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_1t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right]^{p-1}\cdot\left(\lambda_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right)$$

we know that the function $h(t_1,t_2)$ attains its minimum value at point t_1^*, where

$$t_1^*=\frac{(\lambda_1+\lambda_3t_2)\left[\lambda_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right]^{\frac{1}{p-1}}}{\lambda_2^{\frac{1}{p-1}}-\left[\lambda_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right]^{\frac{p}{p-1}}}$$

Noticing that $\left.\frac{\partial h}{\partial t_1}\right|_{t_1=t_1^*}=0$, we have

$$\left[\lambda_1+\lambda_2t_1^*+\lambda_3t_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_1^*t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right]^p=$$
$$\frac{\lambda_2t_1^{*\,p-1}\left[\lambda_1+\lambda_2t_1^*+\lambda_3t_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_1^*t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right]}{\lambda_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_2\delta_X\left(\frac{\varepsilon}{3}\right)}$$

Hence

$$\inf h(t_1,t_2)=h(t_1^*,t_2)=$$
$$\lambda_1+\lambda_2t_1^{*\,p}+\lambda_3t_2^p-$$
$$\left[\lambda_1+\lambda_2t_1^*+\lambda_3t_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_1^*t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right]^p=$$
$$\lambda_1+\lambda_2t_*^p+\lambda_3t_2^p-$$
$$\frac{\lambda_2t_1^{*\,p-1}\left[\lambda_1+\lambda_2t_1^*+\lambda_3t_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_1^*t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right]}{\lambda_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_2\delta_X\left(\frac{\varepsilon}{3}\right)}=$$
$$\lambda_1+\lambda_3t_2^p-\frac{\lambda_2t_1^{*\,p-1}(\lambda_1+\lambda_3t_2)}{\lambda_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_2\delta_X\left(\frac{\varepsilon}{3}\right)}=$$
$$\lambda_1+\lambda_3t_2^p-\frac{\lambda_2(\lambda_1+\lambda_3t_2)^p}{\left(\lambda_2^{\frac{1}{p-1}}-\left[\lambda_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)t_2\delta_X\left(\frac{\varepsilon}{3}\right)\right]^{\frac{p}{p-1}}\right)^{p-1}}$$

Because $h(t_1^*,t_2)$ is strictly increasing in t_2, so we have

$$\inf h(t_1,t_2)\geqslant\lambda_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)^p-\frac{\lambda_2\left(\lambda_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)\right)^p}{\left(\lambda_2^{\frac{1}{p-1}}-\left[\lambda_2-3(\lambda_1\wedge\lambda_2\wedge\lambda_3)\left(1-\frac{\varepsilon}{12}\right)\delta_X\left(\frac{\varepsilon}{3}\right)\right]^{\frac{p}{p-1}}\right)^{p-1}}\geqslant$$
$$\lambda_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)^p-\frac{\left(\lambda_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)\right)^p}{(1-\lambda_2)^{p-1}}=$$
$$f_p^{(2)}(\lambda_1,\lambda_2,\lambda_3,\varepsilon)$$

Hence

$$g(t_1,t_2)\geqslant f_p^{(2)}(\lambda_1,\lambda_2,\lambda_3,\varepsilon)$$

③ If $0<t_1<1-\frac{\varepsilon}{12}<t_2<1$, then

$$g(t_1,t_2)\geqslant\lambda_1+\lambda_2t_1^p+\lambda_3t_2^p-(\lambda_1+\lambda_2t_1+\lambda_3t_2)^p$$

Let $q(t_1,t_2)=\lambda_1+\lambda_2t_1^p+\lambda_3t_2^p-(\lambda_1+\lambda_2t_1+\lambda_3t_2)^p$, then

$$\frac{\partial q}{\partial t_1}=p\lambda_2t_1^{p-1}-p\lambda_2(\lambda_1+\lambda_2t_1+\lambda_3t_2)^{p-1}$$

Since $t_1<t_2$, it follows that $\frac{\partial q}{\partial t_1}<0$, hence

$$g(t_1,t_2)\geqslant\lambda_1+\lambda_2t_1^p+\lambda_3t_2^p-(\lambda_1+\lambda_2t_1+\lambda_3t_2)^p\geqslant$$
$$\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)^p+\lambda_3t_2^p-\left(\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)+\lambda_3t_2\right)^p$$

and

$$q\left(1-\frac{\varepsilon}{12},t_2\right)=\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)^p+\lambda_3t_2^p-\left(\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)+\lambda_3t_2\right)^p$$

From

$$\frac{\partial q\left(1-\frac{\varepsilon}{12},t_2\right)}{\partial t_2}=p\lambda_3t_2^{p-1}-p\lambda_3\left(\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)+\lambda_3t_2\right)^{p-1}$$

we know that the function $q\left(1-\frac{\varepsilon}{12},t_2\right)$ attains its minimum value at point $t_2^*=\frac{\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)}{1-\lambda_3}$.

Hence

$$g(t_1,t_2)\geqslant\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)^p+\lambda_3 t_2^p-\left(\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)+\lambda_3 t_2\right)^p\geqslant$$

$$\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)^p+\lambda_3\left(\frac{\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)}{1-\lambda_3}\right)^p-\left(\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)+\lambda_3\left(\frac{\lambda_1+\lambda_2\left(1-\frac{\varepsilon}{12}\right)}{1-\lambda_3}\right)\right)^p=$$

$$f_p^{(3)}(\lambda_1,\lambda_2,\lambda_3,\varepsilon)$$

④ If $0<t_2<1-\frac{\varepsilon}{12}<t_1<1$, then, using the similar method which is used in proof of case ③, we can deduce that

$$g(t_1,t_2)\geqslant\lambda_1+\lambda_2 t_1^p+\lambda_3 t_2^p-(\lambda_1+\lambda_2 t_1+\lambda_3 t_2)^p\geqslant$$

$$\lambda_1+\lambda_2 t_1^p+\lambda_3\left(1-\frac{\varepsilon}{12}\right)^p-\left(\lambda_1+\lambda_2 t_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)\right)^p\geqslant$$

$$\lambda_1+\lambda_2\left(\frac{\lambda_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)}{1-\lambda_2}\right)^p+\lambda_3\left(1-\frac{\varepsilon}{12}\right)^p-\left(\lambda_1+\lambda_2\left(\frac{\lambda_1+\lambda_3\left(1-\frac{\varepsilon}{12}\right)}{1-\lambda_2}\right)+\lambda_3\left(1-\frac{\varepsilon}{12}\right)\right)^p=$$

$$f_p^{(4)}(\lambda_1,\lambda_2,\lambda_3,\varepsilon)$$

Hence

$$g(t_1,t_2)\geqslant\min\{f_p^{(1)},f_p^{(2)},f_p^{(3)},f_p^{(4)}\}=\varphi_p(\lambda_1,\lambda_2,\lambda_3,\varepsilon)$$

Combining these inequalities with (11), we have that

$$\|\lambda_1x_1+\lambda_2x_2+\lambda_3x_3\|^p+\varphi_p(\lambda_1,\lambda_2,\lambda_3,\varepsilon)\leqslant\lambda_1\|x_1\|^p+\lambda_2\|x_2\|^p+\lambda_3\|x_3\|^p$$

for

$$\forall x_1\in S(X)\text{ and }x_2,x_3\in U(X)$$

Let

$$\delta_p(\lambda_1,\lambda_2,\lambda_3,\varepsilon)=\min\{\varphi_p(\lambda_1,\lambda_2,\lambda_3,\varepsilon),\varphi_p(\lambda_2,\lambda_1,\lambda_3,\varepsilon),\varphi_p(\lambda_3,\lambda_2,\lambda_1,\varepsilon)\}$$

then, for each $p\in(0,1)$, there exists a strictly increasing function $\delta_p(\lambda_1,\lambda_2,\lambda_3,\cdot):\mathbf{R}^+\to\mathbf{R}^+$, $\delta_p(\lambda_1,\lambda_2,\lambda_3,0)=0$, such that

$$\left\|\sum_{i=1}^3\lambda_i x_i\right\|^p+(\|x_1\|\vee\|x_2\|\vee\|x_3\|)^p\delta_p\left(\lambda_1,\lambda_2,\lambda_3,\frac{A(x_1,x_2,x_3)}{(\|x_1\|\vee\|x_2\|\vee\|x_3\|)^2}\right)\leqslant\sum_{i=1}^3\lambda_i\|x_i\|^p,\ \forall x_1,x_2,x_3\in X$$

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